Math 345A Class Exercise: Assessment

Dr. Fred Park, Whittier College

A differential equation is an equation that involves an unknown function and its derivatives. They occur in almost any real world application. One simple example is in population growth. Let $p(t)$ be the population of a colony of animals at time $t > 0$. If we assume that the rate of change of the population is proportional to the square of the present population for example, we arrive at the following equation:

$$\frac{dp(t)}{dt} = kp(t)^2$$

The above equation is an example of a differential equation where the unknown function $p(t)$ is the population at time t, and the equation involves $p(t)$ and its derivative dp/dt.

Try to answer the questions below to the best of your ability.

1. Suppose a given animal population grows at a rate proportional to its population. Write out a differential equation modeling this phenomenon. If the initial population is 500, what is the population at time $t = 10$? What is the explicit population for all time t? At what time t will the population double from its initial one?

2. Suppose that a population grows at a rate that is proportional to its population but at the same time, if the population becomes too large, the growth begins to decrease. Write down an explicit differential equation modeling this phenomenon. Draw a graph of typical solutions.

3. Can you write down a piecewise continuous function that models the following population dynamic. It begins with 500 people, grows exponentially, when t=5 years, you have 3500 people, but when t=10 years, the population becomes constant. Graph your result. Can this be a solution to a differential equation?

4. **Applied Challenge Problem:** Water flows from an inverted tank with circular orifice at the rate:

$$\frac{dx}{dt} = -0.6\pi r^2 \sqrt{2g}\frac{\sqrt{x}}{A(x)}$$

where r is the radius of the orifice, x is the height of the liquid level from the vertex of the cone, and $A(x)$ is the area of the cross section of the tank x units above the orifice. Suppose $r = 0.1$ ft, g $=$ 32.1 ft/s^2, and the tank has an initial water level of 8 ft and initial volume of $512(\pi/3)\text{ft}^3$. Find the following:

   (a) The water level after 10 min with h= 20 s.

   (b) When the tank will be empty, to within 1 min.