Math 354, Class Exercise 11
Lotka-Volterra Equations
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(This is a closed book active learning exercise!) Consider a 2 species model of predator and prey. To simplify, let us consider the following two species in a closed surrounding:

F = number of a certain species of fish (eaten by sharks) like tuna, mackerel, or sardines for example in a specific region of the sea.

S = number of sharks in the same area

The area is assumed to be bounded with no migration across the boundary of the region. Then the rates of change of the population of the fish and sharks depends on the number of fish and sharks present. That is: We found the following model to hold:

$$\frac{dF}{dt} = F(a - bF - cS) \tag{1}$$

$$\frac{dS}{dt} = S(-k + \lambda F) \tag{2}$$

Which are called the *Lotka-Volterra Equations*. If we assume $b = 0$, i.e. Non-Logistic fish growth, then the equations reduce to:

$$\frac{dF}{dt} = F(a - cS) \tag{3}$$

$$\frac{dS}{dt} = S(-k + \lambda F) \tag{4}$$

(a) Find the equilibrium solutions to the above equations.

(b) Look at isoclines by analyzing dF/dS in the S-F phase plane. (S sill be x-axis and F the y-axis).

(c) Draw arrows indicating the flow direction of the solution curve on the isoclines and also on the equilibrium lines.

(d) Near the non-trivial equilibrium, what is the local behavior of the trajectories? Explain by looking at rates of increase or decrease. i.e. $dF/dt > 0$ when $S < a/c$ etc.

(e) Near the non-trivial equilibrium, do you get a spiral? a close path? what are the possible cases?

(f) What does your intuition tell you about the trajectories near an equilibrium point?

1. Do as indicated below:

   (a) Linearize the differential equation near the equilibrium population by setting F and S equal to the following:

   $$F = \frac{k}{\lambda} + \epsilon F_1 \tag{5}$$

   $$S = \frac{a}{c} + \epsilon S_1 \tag{6}$$

   (b) Reduce your first order system of 2 equations into a single second order equation by eliminating F_1 from the second equation above.

(c) Write down a solution of the second order equation above. It should be of the form

$$S_1 = S_{10}\cos(\omega t) + Q\sin(\omega t)$$

with $\omega = (ak)^{1/2}$ the circular frequency.

(d) What is the solution F_1?

(e) What is the period of the oscillation in both solutions?

(f) Show that in the vicinity of the equilibrium population, there is a "constant of motion" specified by

$$\left(\frac{dS_1}{dt}\right)^2 + akS_1^2 = \text{constant}$$

(g) Show that the constant of motion condition above yields an equation of an ellipse in the $S_1 - F_1$ phase plane. Draw the picture now with isoclines etc. How can you explain this periodic solution and the relationship between F_1 and S_1 with regard to time t?

2. Consider the predator prey model with b=0 in question 1 above. Suppose the value of k was increased

   (a) what does increasing the value of k correspond to ecologically?

   (b) How does this affect the non zero equilibrium population?

   (c) briefly explain this effect ecologically?

3. Consider the predator prey model question 1 above and consider the case when $b \neq 0$. Calculate all possible equilibrium solutions. Compare these to the case when $b = 0$. Briefly explain the quantitative and qualitative differences between these two cases?