Math 354, Class Exercise 11
Gradient Descent
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1. Let $z = f(x, y)$ be a function defined for $(x, y) \in \Omega$, with Ω a domain. What direction does the gradient vector: ∇f point in relation to the aforementioned surface? Prove that the directional derivative $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ is maximized when $\vec{u}$ and ∇f point in the same direction. When is it minimized?

2. Apply gradient descent starting with $x_0 = 5$ for the function $f(x) = x^2$ on [-5,5]. Do the same for $x_0 = -5$. What do you notice? Does the algorithm work?

3. Apply gradient descent to the surface $z = f(x, y) := (x/2)^2 + (y/5)^2 + \pi$, starting with $(x_0, y_0) = (5, 7)$. Where does it converge to? Is this reasonable?

4. Consider the Rosenbrock function

$$f(x, y) = (a - x)^2 + b(y - x^2)^2. \tag{1}$$

   If we set $a = 1$ and $b = 100$, we obtain the function:

$$f(x, y) = (1 - x)^2 + 100(y - x^2)^2.$$

   (a) On the domain $\Omega = [-2, 2] \times [-1, 3]$, plot the Rosenbrock function as a surface in $\mathbb{R}^3$.

   (b) Using $x_0 = -2$ and $y_0 = 2$, apply gradient descent to find a min of the function.

   (c) Use calculus techniques to prove that the Rosenbrock function has a global min at (a, a^2) where $f(x, y) = 0$.