Math 354, Class Exercise 3
Logistic Equation, Phase Planes, and Perturbation Analysis
Active Learning Approach Instructor: Dr. Fred Park

1. Given the logistic equation $\frac{dN}{dt} = N(a - bN)$, sketch the 1D phase plane and typical solutions if you haven't already done so. Most of you have though.

2. Plot a sketch of $\frac{dN}{dt}$ vs N with $\frac{dN}{dt}$ being the y-axis. Place arrows on the curve depending on the direction of where the slopes are converging. Can you make the connection between typical solutions and the 1D phase plane to the 2D diagram?

3. The equilibrium solution $N = a/b$ is also called the Saturation Level. Show that solution of the logistic equation has an inflection point at a population equal to 1/2 the saturation level. You should be able to do this in 1 minute flat.

4. (Linear Stability Analysis) Use Taylor series to show that the non-zero equilibrium population of the logistic equation is stable. Show that if the initial population is near but not at the equilibrium population, the population does not ever attain the equilibrium population in finite time. To begin, you should consider a solution of the form:

$$N(t) = \frac{a}{b} + \epsilon N_\epsilon(t) + O(\epsilon^2)$$

   where ϵ is chosen small enough so that $|\epsilon N_\epsilon| << \frac{a}{b}$.

5. (a) Consider $dN/dt = \alpha N^2 - \beta N$ with $\alpha, \beta > 0$.
   (b) How does the growth rate depend on the population?
   (c) Sketch the solution in the phase plane labeling all equilibria. Sketch typical solutions.
   (d) Are there any inflection points in the solution assuming sufficient smoothness of solutions?
   (e) Obtain the exact solution.
   (f) Show how parts (b) and (c) illustrate the following behavior:
      i. If $N_0 > \beta/\alpha$, then $N \to \infty$. At what time does $N \to \infty$?
      ii. If $N_0 < \beta/\alpha$, then $N \to 0$.
      iii. What happens if $N_0 = \beta/\alpha$?