Math 354, Class Exercise 6
The Logistic Equation and some of its Variants
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1. **Time step restrictions for Euler's Method**
   Consider the Logistic Equation:
$$\frac{dN}{dt} = N(a - bN). \tag{1}$$
   Here, we consider the case where $a = 2$ and $b = 1$.
   (a) Run the script `euler_logistic_script.m` with different choices of Δt. In the script $\Delta t = dt$. Note, you can change dt by adjusting M which is the number of steps. They are inversely proportional to each other. Why? A good starting place for experimenting with time steps would be using the values $M =$ 15, 25, 35, 45, 55. What do you notice? What were the corresponding time steps? Take note of them.
   (b) Calculate a restriction bound on the time step Δt in Euler's method by assuming that the $\mathcal{O}(\Delta t^2)$ term is much less than the $\mathcal{O}(\Delta t)$ term from the derivation. Note, you should be able to get an actual number if we assume $N < 5$. This is reasonable from a phase plane stability analysis assuming that one of the cases has $N_0 = 5$ for this problem and the other has $N_0 = 0.1$.
   (c) Does the time step restriction on Δt you found in (b) allow for the convergence of Euler's Method in this context? Experiment by running `euler_logistic_script.m` with values of Δt larger and smaller than this estimated step size restriction. What do you notice?

2. **Direction Fields**
   Consider the variant of the Logistic Equation
$$\frac{dN}{dt} = -\alpha N + \beta N^2 \tag{2}$$
   Let us consider the case when $\alpha = 2$ and $\beta = 1$.
   (a) The right hand side of equation (2) gives the slope of the tangent line to the solution for a corresponding N value. If we plot all pieces of those tangent lines in the $t - N$ plane, we obtain a direction field (as seen from the demo in class). Modify the script `direction_field_logistic.m` to plot the direction field for equation (2).
   (b) What can you say about solutions to equation (2) that have initial values $N_0 > 2$? What if $N_0 < 2$? Does your intuition tell you that there is a finite time blowup of solutions for certain initial values?

3. **Linear Stability Analysis Revisited (class favorite!)**
   (a) Using linear stability analysis, what can you deduce about the stability of the $N = 0$ equilibrium in equation (2)? Here, consider perturbations from the zero equilibrium of the form $N(t) = 0 + \epsilon N_\epsilon(t)$ where $|\epsilon N_\epsilon(t)|$ is sufficiently small.
   (b) What can you deduce about the stability of the $N = \alpha/\beta$ equilibrium?
   (c) Using the script `euler_logistic_script.m`, modify it to approximate solutions to equation (2) using initial values $N_0 = 2 - 10^{-5}$ and $N_0 = 1$. Here, $t_0 = 0$ and $t_{final} = 10$ and $M = 100$ steps. Does the computation agree with your stability analysis? Lastly, what happens if you run the code for $N_0 = 2 + 10^{-5}$? Explain what the issue is?

4. **Logistic Equation with Harvesting**
Consider the Modified Logistic Equation

$$\frac{dN}{dt} = N(a - bN) - H \tag{3}$$

where H is a constant and we are under the assumption that members of the population are leaving at a constant rate. Let $a, b, H > 0$.

(a) In words, what effect do you think H has on the equilibriumm ($\longleftarrow$ typo) populations?

(b) If $a = 2$, $b = 1$, and $H = 1/2$, find the equilibrium populations?

(c) Run your code for this particular case using $N_0 = 5$ and $N_0 = 1$. What do you notice? What happens if $H = N_0$? What if $H < N_0$?