Math 354, Class Exercise 8
Phase Plane Fun
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1. Consider the system

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x}$$

where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Points in the $x_1 - x_2$ plane correspond to solutions of the above DE. The $x_1 - x_2$ plane is called the Phase Plane.

(a) Method of Isoclines. Choose points in the phase plane and calculate the derivative of solutions near the point by using the DE above. Then graph the vector obtained from the derivative emanating from the point. Trace a curve at a given starting point in the direction of the derivative. This piece of curve is called an isocline. A couple good ones to try would be:

$$\vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

What does this tell us about the qualitative behavior of the solutions in the phase plane?

(b) Are there lines in the phase plane where the isoclines have slope zero or ∞? Can you find any for the above system?

(c) Solve the initial value problem:

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

(d) What are the eigenvectors associated to the solution of the above problem? Graph the two lines that are in the direction of each eigenvector $\vec{v}_1$ and $\vec{v}_2$.

(e) For the solution corresponding to the first eigenvector and eigenvalue, for increasing values of t where does the solution tend in the phase plane? If it goes towards the origin, then on the associated line, draw arrows indicating the movement towards the origin. If it moves away, then draw them away. Doe ($\longleftarrow$ typo) this for the second solution as well.

(f) Your lines should break the phase plane into 4 quadrants. Draw typical solution trajectories for increasing values of t in each quadrant and explain why they look the way they do? Are they tangent to the eigenvector lines?

2. Sketch the phase plane solution for the following system. Make sure to explain all qualitative behavior.

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$$

3. Sketch the phase plane solution for the following system. Make sure to explain all qualitative behavior.

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix} \vec{x}$$