Math 354, Class Exercise 9
More Phase Plane Fun
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1. What do points in the phase plane represent? Also, what are the equilibria? What do the trajectories represent in the phase plane?

2. Sketch the phase plane solution for the following system. Make sure to explain all qualitative behavior.

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$$

3. Sketch the phase plane for the following system

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix} \vec{x}$$

Here, we assume that the eigenvalues are

$$r_{1,2} = \pm 3\sqrt{3}i$$

The first eigenvector corresponding to e-value $r_1 = 3\sqrt{3}i$ is

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 1 - \sqrt{3}i \end{pmatrix}$$

the second eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 3 \\ 1 + \sqrt{3}i \end{pmatrix}$$

The solution we obtain from the first e-vector is

$$\vec{x}_1(t) = e^{3\sqrt{3}i\,t} \begin{pmatrix} 3 \\ 1 - \sqrt{3}i \end{pmatrix}$$

using Euler's formula we obtain:

$$\vec{x}_1(t) = \begin{pmatrix} 3\cos(3\sqrt{3}t) + 3i\sin(3\sqrt{3}t) \\ \cos(3\sqrt{3}t) + i\sin(3\sqrt{3}t) - \sqrt{3}i\cos(3\sqrt{3}t) + \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} 3\cos(3\sqrt{3}t) \\ \cos(3\sqrt{3}t) + \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} + i \begin{pmatrix} 3\sin(3\sqrt{3}t) \\ \sin(3\sqrt{3}t) - \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix} \tag{2}$$

$$= \vec{u}(t) + i\,\vec{v}(t) \tag{3}$$

where $\vec{u}(t)$ and $\vec{v}(t)$ are both linearly independent solutions to the original problem. Show that any solutions of this form are either circular or elliptical. How can you determine the orientation of the curves. i.e. which way the trajectories are moving in time.

4. Sketch the phase plane solution for the following system. Make sure to explain all qualitative behavior.

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \vec{x}$$

Here, we assume that the eigenvalues are

$$r_{1,2} = 2 \pm 8i$$

The general solution is

$$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 8\cos(8t) + \sin(8t) \\ \sin(8t) \end{pmatrix} \tag{4}$$