

# Green's First Identity

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## 1 Green's First Identity for Variational Problems

The context will be image processing variational problems. Here, we seek to minimize a functional of the form:  $J[u]$  where  $u = u(x, y)$  defined on an image domain  $\Omega$ . Let  $\partial\Omega$  denote the boundary of the image domain. Then the following holds:

$$\iint_{\Omega} \nabla f \cdot \nabla g \, dA = - \iint_{\Omega} (\nabla \cdot \nabla f) g \, dA + \int_{\partial\Omega} g (\nabla f \cdot \vec{n}) \, ds \quad (1)$$

Note that this is basically multi-variable integration by parts. The boundary terms yields a condition on one of the functions when in the context of variational minimization. Note that:

$$\frac{\partial f}{\partial \vec{n}} = \nabla f \cdot \vec{n} \quad (2)$$

is the directional derivative of  $f$  in the normal direction to the boundary.