Math 354: Mathematical Modeling Midterm

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Thursday October 24th, 2019
Due Monday October 28th 11:59pm

Name:____

Student ID#:____

Note: For full credit you must show all work. Incoherent work without logic or reason will not receive any credit whatsoever! This exam is individual work only! **Only your notes and textbook are to be used as resources.** For the computing portion, you may only use the Matlab documentation or help command. No other external resources can be used.

1. Consider the following system:

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = 1 - x^2$$

   (a) Calculate the equilibrium solutions.

   (b) Linearize the above system about the equilibrium solutions.

   (c) Using the eigenvalue/eigenvector technique from class, sketch the phase plane that includes both equilibria and typical trajectories. Note: solution trajectories need to be qualitatively correct to obtain any credit.

2. You may have noticed a rise in the coyote population around campus here at Whittier. You may have also seen quite a few rabbits as well. The coyotes are predators and in fact prey on the rabbits. We model the interaction between these two species in the following manner. Consider the Lotka-Volterra equations that govern a predator-prey system:

$$\begin{aligned}\frac{dR}{dt} &= R(a - bR - cC)\\ \frac{dC}{dt} &= C(-k + \lambda R)\end{aligned} \tag{1}$$

   with $b \neq 0$. Here a, b, c, k, and λ are all constants. $C = C(t)$ is the number of coyotes at time t and $R = R(t)$ the number of rabbits at time t.

   (a) If the number of coyotes C is always zero, what does the system (1) reduce to exactly? What is the model now called?

   (b) If the number of rabbits R is always zero, what does the system (1) now reduce to exactly? What's it called?

   (c) Consider the two equilibrium populations:

$$R = \frac{k}{\lambda} \quad \text{and} \quad C = \frac{a}{c} - \frac{bk}{c\lambda} > 0$$

   associated to (1). Write out the constant coefficient linear system of differential equations that governs the small displacements from the above equilibrium population using Taylor series for a function of two variables. See equation 44.5 in the book for the formulation of the 2D Taylor expansion or my notes from class.

(d) Show that your answer to (a) and (b) can be put in the form:

$$\frac{dR_1}{dt} = \frac{k}{\lambda}(-bR_1 - cC_1) \tag{2}$$
$$\frac{dC_1}{dt} = \left(\frac{a}{c} - \frac{bk}{c\lambda}\right)\lambda R_1 \tag{3}$$

where R_1 and C_1 are displacements from the equilibrium populations of the rabbits and coyotes respectively. Eliminate R_1 from the second equation to obtain a constant coefficient second order differential equation for C_1. Analyze the equation and determine the conditions under which the equilibrium population is stable.

3. Real world modeling of Coyote and Rabbits on campus. Consider the Lotka-Volterra equations that govern a coyote–rabbits (predator–prey) system below:

$$\begin{aligned}\frac{dR}{dt} &= R(0.16 - 0.004C)\\ \frac{dC}{dt} &= -C(1.2 - 0.02R).\end{aligned} \tag{4}$$

$C = C(t)$ is the number of coyotes at time t and $R = R(t)$ the number of rabbits at time t.

(a) Find the non-zero equilibrium populations.

(b) Linearize the system about the non-zero equilibrium point.

(c) Without calculating the eigenvectors to the linearized system, can you say anything about the qualitative behavior of solutions in the phase plane? Explain clearly but do not compute the eigenvectors!

(d) Using Matlab and forward Euler, compute the solutions for 50 years with time-step of dt = 0.0001 assuming that $R(0) = 60$ and $C(0) = 100$.

Here, forward Euler is the numerical approximation scheme from Math 345A. For example, if one has a differential equation $dy/dt = f(y, t)$ then one can approximate the solution at a given time t_N. Here, $t_n = t_0 + \text{ndt}$ and $y_0 = y(t_0)$ is the initial value. y_N is the approximation to the solution $y(t_N)$ at t_N. Euler's method for this first order DE is the following:

$$y_{n+1} = y_n - (t_{n+1} - t_n)f(t_n, y_n) \tag{5}$$
$$= y_n - \text{dt}f(t_n, y_n) \quad \text{for } n = 0, 1, \ldots, N-1. \tag{6}$$

You will extend the above method to the two dimensional system (very straightforward).

(e) Plot both populations of the coyotes and rabbits vs time on the same graph. Clearly label the graph.

(f) Plot the rabbits (on x-axis) and the coyotes (on y-axis) on the same graph without time. Clearly label the graph. Note, this is the phase plane solution but we numerically calculate it. What do you notice?

(g) Are the populations cyclic? Which way do they cycle? Can you show qualitatively what the direction for increasing t is in the phase plane using the equations above (4)?

4. Let us consider the case when two species are competing with each other where the relation is not predator-prey. One way to model this would be to consider logistic growth in the population of both species. We assume the effect of the competition is to reduce each others growth respectively. Therefore, we can assume each species will reduce the other species' growth rate proportional to their population. Let us look at the specific example of two Salmon species x and y respectively:

$$\frac{dx}{dt} = x(180 - 2x - y)$$
$$\frac{dy}{dt} = y(120 - 2y - x)$$

(a) Find all equilibrium solutions. Explain what each one signifies in relation to the physical salmon competition model.

(b) Using Matlab, plot the direction field and equilibrium solutions on the same graph.

(c) Based on your direction field, what are the stable equilibrium populations? What is the physical relevance of them?