

MATH 241: Practice Final

Dr. Fred Park

Note: this exam is only slightly longer than the actual final exam.

1. True/False. If true, show why. If false, find an explicit counter example.
 - (a) If $f \in C^1$ on \mathbb{R}^3 and C is any circle, then $\int_C \nabla f \cdot d\vec{r} = 0$?
 - (b) If $f \in C^\infty$ meaning it has continuous partial derivatives of all orders on \mathbb{R}^3 , then $\text{div}(\text{curl}\vec{F}) = 0$?
 - (c) If \vec{F} is a vector field, then $\text{div}\vec{F}$ is also a vector field?
 - (d) If \vec{F} is a vector field, then $\text{curl}\vec{F}$ is also a vector field?
 - (e) Let C be a given plane and $-C$ be the same curve but with the orientation reversed. $\int_{-C} f(x, y)ds = -\int_C f(x, y)ds$.
2. Evaluate the line integral $\int_C yz \cos x ds$ with C defined by: $x = t$, $y = 3 \cos t$, $z = 3 \sin t$, $0 \leq t \leq \pi$.
3. Show that \vec{F} is a conservative vector field then find a function f such that $\nabla \cdot f = \vec{F}$ where $\vec{F} = (1 + xy)e^{xy}\vec{i} + (e^y + x^2e^{xy})\vec{j}$.
4. Evaluate the surface integral $\int_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle x, y, 5 \rangle$ and S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$
5. (a) Let C be a simple closed piecewise-smooth space curve which lies entirely in a plane and suppose that the plane has upward pointing unit normal vector given by $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Show that the area of the portion of the plane enclosed by C is:

$$\frac{1}{2} \int_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz$$

- (b) Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented counter-clockwise as viewed from above. Show that the

integral

$$\oint_C 2ydx + 3zdy - xdz$$

depends only on the area of the region enclosed by C and not on the position or shape of C .

6. Let \mathbf{F} be a vector field defined on $\mathbb{R}^2 - \mathbf{0}$ defined by:

$$\mathbf{F}(x, y) = \frac{1}{x^2 + y^2} \langle -y, x \rangle$$

(a) Show that for any positively-oriented simple closed curve C in \mathbb{R}^2 which encloses the origin,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$$

(b) Let C be a positively oriented simple closed curve in \mathbb{R}^2 which doesn't pass through or enclose the origin. Show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

7. Let $\mathbf{F}(x, y, z) = \langle e^{yz}, xze^{yz} + z \cos y, xye^{yz} + \sin y \rangle$ be a vector field.

(a) Let C be the piecewise smooth path which travels in a straight line from $(1, 0, 1)$ to $(1, 0, 0)$, then along the x-axis from $(1, 0, 0)$ to $(0, 0, 0)$, then finally from $(0, 0, 0)$ to $(1, \pi/2, 0)$ along the parabola specified by: $y = \frac{\pi x^2}{2}$ and $z = 0$. Compute:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

(b) Does there exist a vector field \mathbf{G} such that $\mathbf{F} = \text{curl}(\mathbf{G})$? Why or why not?

8. Let $\mathbf{F} = \langle yz^3, yz^2, x^7 e^{xy^2} \rangle$ be a vector field. Let E denote the region defined by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and S be the boundary of E with outward unit normal given by \mathbf{n} and a, b, c are all constants. Compute the Flux:

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$