MATH 241: Practice Final

Fall 2014 – Dr. Fred Park

Note: this exam is only slightly longer than the actual final exam.

1. True/False. If true, show why. If false, find an explicit counter example.

   (a) If $f \in C^1$ on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\vec{r} = 0$?

   (b) If $f \in C^\infty$ meaning it has continuous partial derivatives of all orders on $\mathbb{R}^3$, then $div(curl\vec{F}) = 0$?

   (c) If $\vec{F}$ is a vector field, then $div\vec{F}$ is also a vector field?

   (d) If $\vec{F}$ is a vector field, then $curl\vec{F}$ is also a vector field?

   (e) Let C be a given plane and $-C$ be the same curve but with the orientation reversed. $\int_{-C} f(x,y)ds = -\int_C f(x,y)ds$.

2. Evaluate the line integral $\int_C yz\cos x ds$ with C defined by: $x = t$, $y = 3\cos t$, $z = 3\sin t$, $0 \leq t \leq \pi$.

3. Show that $\vec{F}$ is a conservative vector field then find a function f such that $\nabla \cdot f = \vec{F}$ where $\vec{F} = (1+xy)e^{xy}\vec{i} + (e^y + x^2e^{xy})\vec{j}$.

4. Evaluate the surface integral $\int_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle x, y, 5\rangle$ and S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

5. (a) Let C be a simple closed piecewise-smooth space curve which lies entirely in a plane and suppose that the plane has upward pointing unit normal vector given by $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Show that the area of the portion of the plane enclosed by C is:

$$\frac{1}{2}\int_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz$$

   (b) Let C be a simple closed smooth curve in the plane $2x+2y+z = 2$, oriented counter-clockwise as viewed from above. Show that the integral

$$\oint_C 2ydx + 3zdy - xdz$$

depends only on the area of the region enclosed by C and not on the position or shape of C.

6. Let $\mathbf{F}$ be a vector field defined on $\mathbb{R}^2 - \mathbf{0}$ defined by:

$$\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}\langle -y,\, x\rangle$$

   (a) Show that for any positively-oriented simple closed curve C in $\mathbb{R}^2$ which encloses the origin,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$$

   (b) Let C be a positively oriented simple closed curve in $\mathbb{R}^2$ which doesn't pass through or enclose the origin. Show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

7. Let $\mathbf{F}(x, y, z) = \langle e^{yz},\, xze^{yz} + z\cos y,\, xye^{yz} + \sin y\rangle$ be a vector field.

   (a) Let C be the piecewise smooth path which travels in a straight line from (1, 0, 1) to (1, 0, 0), then along the x-axis from (1, 0, 0) to (0, 0, 0), then finally from (0, 0, 0) to $(1, \pi/2, 0)$ along the parabola specified by: $y = \frac{\pi x^2}{2}$ and $z = 0$. Compute:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

(b) Does there exist a vector field $\mathbf{G}$ such that $\mathbf{F} = curl(\mathbf{G})$? Why or why not?

8. Let $\mathbf{F} = \langle yz^3,\, yz^2,\, x^7 e^{xy^2} \rangle$ be a vector field. Let E denote the region defined by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and S be the boundary of E with outward unit normal given by $\mathbf{n}$ and a, b, c are all constants. Compute the Flux:

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$